Active Epistemics: Methods, Recommendations and Hypotheses to Learn

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ABSTRACT. Modal operator theory is initially reviewed as a multimodal framework in which active agents, their behavior and environment may be modeled and studied. The intersection between agency, learnability and temporally fluctuating hypotheses is then scrutinized from an epistemological perspective.

Recent years have witnessed impressive leaps forward in modelling the dynamics of inquiry. Different models have been suggested to fathom various aspects of the reasoning processes relevant to inquiry; belief revision, dynamic epistemic and doxastic logics, stit-theory, logics of action, intention and doxastic voluntarism—in general, logics and formal models for – using an adequately covering term coined by Parikh – *social software* [Par02]. Many of these logics and formal frameworks rely on combining different modal logics in the realization that inquiry incorporates a multiplicity of parameters including time, change, relevant alternatives, epistemological forcing and learning just to mention a few. The result: *Multi-modal systems.*

A particular multi-modal system called *modal operator theory* is in focus here [Hen01], [HP03], [Hen06], [HP09]. It was developed to study the properties of limiting convergent knowledge by combining epistemic, tense and alethic logic with rudimentary elements from formal learning theory or computational epistemology [Kel96]. The paper is largely expository and includes

- an introduction to modal operator theory and its formal framework,
- a presentation of some key results,
- some new results pertaining to learnability, methodology and the temporal behavior of hypotheses.[1]

[1] Most proofs are omitted in this presentation, but readers are referred to [Hen01] and [HP09] for more details.

1 A Model of Inquiry

Assume that an agent interacts with the world through evidence streams: An evidence stream ε is an ω-sequence of natural numbers, *i . e.,* $\varepsilon \in \omega^\omega$. Hence, an evidence stream has the form $\varepsilon = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_n, ...)$ and consists of code numbers of evidence: At each stage i in inquiry ε_i is the code number of all evidence acquired at this stage. Evidence streams can encode all kinds of discrete evidence of both observational and/or theoretical nature and thus evidence streams are *not* data streams. Evidence is whatever can eventually figure in a scientific report construed in the broadest possible sense.

A possible world is a pair consisting of an evidence stream ε associated with a state coordinate n. A possible world has the form (ε, n) such that $\varepsilon \in \omega^\omega$ and $n \in \omega$. The set of all possible worlds is $\mathcal{W} = \{(\varepsilon, n) \mid \varepsilon \in \omega^\omega, n \in \omega\}$.

Let $\varepsilon \mid n$ denote the finite initial segment of evidence stream ε of length n. One may think of $\varepsilon \mid n$ as the evidence seen up to state n, *i. e.* $\varepsilon \mid n = a_0, a_1, ...a_{n-1}$ The rest $\varepsilon \backslash \varepsilon \mid n = a_n, a_{n+1}, ...,$ is the evidence that one would observe if the world develops according to (ε, n). Define $\omega^{<\omega}$ to be the set of all finite initial segments of elements in ω. Let $(\varepsilon \mid n)$ denote the set of all infinite evidence streams that extends $\varepsilon \mid n$. Refer to the finite initial segment $\varepsilon \mid n$ as the **handle** with **fan** $(\varepsilon \mid n)$. Let the **world-fan** be defined as $[\varepsilon \mid n] = (\varepsilon \mid n) \times \omega$,*i. e.* $[\varepsilon \mid n] = \{(\tau, k) \mid k \in \omega \text{ and } \tau \mid k = \varepsilon \mid n\}$. An element in (τ, k) in $[\varepsilon \mid n]$ is a possible world where $\tau \mid k = \varepsilon \mid n$ if $k \leq n$, and $\tau \mid n = \varepsilon \mid n$ if $k \geq n$. That is, if k is some later time,$k > n$, (τ, k) is a possible extension of the world (ε, n) up to time k.

The world fan is the background knowledge over which the method has to force in order to obtain knowledge. Note that the background knowledge concentrates ever tighter as times goes by. Refer to this property as the **shrinking property** (SP) of the background knowledge.

LEMMA 1. *The Shrinking Property*

$$...[\varepsilon \mid n+k] \subset ... \subset [\varepsilon \mid n+1] \subset [\varepsilon \mid n]. \qquad (SP)$$

Every later background knowledge is properly included in every earlier background knowledge.

The characterization of evidence streams, state coordinates and possible worlds impose a branching time structure.

The current structure is closely related to standard Ockhamistic semantics. An Ockhamistic tense structure is typically given by:

1. *O is a non-empty set of moments in time,*

2. *$<$ is an irreflexive and transitive earlier-later relation which is backwards linear.*

The concept of *chronicles* c_{Ock} plays an important role in Ockhamistic semantics; they may be understood as possible courses of events or evidence and are individually defined as being maximal linear subsets of structure $(O, \prec)$. In general, define the Ockhamistic *model* to be a triple $(O, \prec, V_{Ock})$ where V_{Ock} is a valuation function which assigns truth-values $V_{Ock}(n, a)$ to all pairs where the first argument is an element of O and the second argument a a propositional variable.

It is not too surprising that the current tense structure is branching. In the course of inquiry the agent is shown the world in a piecemeal fashion. The agent is exposed to the world course by observing ever increasing segments of evidence. For every 'now' the method has only been shown some finite chunk of the world and is eventually asked to perform a conjecture pertaining to the world's infinite trajectory. This is exactly where the problem of induction lies. The world may take any of the uncountable many courses specified by the background knowledge for every later 'now'. Thus, from some specified 'now' onwards, time and event courses are branching for all the methods knows. Now based on the Ockhamistic tense structure define the current tense structure accordingly: The tense structure is pair $(\mathbb{T},\prec)$ such that $\mathbb{T} = \{\varepsilon \mid n \mid \varepsilon \in \omega^\omega, n \in \omega\}$ and $t_1 \prec t_2$ *iff* $\varepsilon \mid n = \tau \mid n$ *and* $n < m$. The evidence stream ε is the actual (privileged) evidence stream. The state coordinate n, for some specified n, is to be thought of as the "age" of the world (ε, n), *i. e.*, n is the time in the branch (or the universal clock). The earlier-later relation is defined with respect to finite initial segments.

A *moment* in time t is the branching moment of times. In other words, let a moment in time be a finite initial segment $t \in (\mathbb{T}, \prec)$. Hence a moment in time is defined as the course of events up until 'now'. Time is based on events and hence the branching time structure is given by events. In other words we are working on the entire tree of finite sequences of natural numbers (the full Brouwerian spread over the natural numbers) where finite initial segments (of events) defines moments in time.

It is obvious that $(\mathbb{T},\prec)$ satisfies the above two requirements; the relation is clearly irreflexive, transitive and backwards linearly ordered. There exist, of course, many other Ockhamistic tense structures. One example would be a structure that extends into the indefinite past. The current structure, however, has a starting point. A chronicle c may be defined as a maximal linear subset of $(\mathbb{T},\prec)$ with the form $c_\varepsilon = \{\varepsilon \mid k : k \in \omega\}$ for some $\varepsilon \in \omega^\omega$. Finally the set of all chronicles is $\mathcal{C} = \{c_\varepsilon : \varepsilon \in \omega^\omega\}$.

2 Methods

Given the above characterization one can define an agent as a discovery method accordingly: *A* discovery method δ *is a function from finite initial* *segments of evidence to hypotheses*, *i. e.*: $\delta : \omega^{<\omega} \longrightarrow \mathcal{H}$. This definition does not prevent the discovery method from conjecturing absurdities. It is therefore assumed that no discovery methods of interest are allowed to conjecture absurdities. The discovery methods are non-trivial in the sense that $\forall(\varepsilon, n) : \delta(\varepsilon \mid n) \neq \emptyset$.

At some point the method performs a conjecture and stays with it. The method is not required to discover exactly some hypothesis of interest. It suffices that the method produces a conjecture which continues to entail the hypothesis in question. Limiting convergence and identification then simply means that there exists a time n such that for each later time n' the method produces a consistent conjecture entailing the hypothesis h: δ discovers h on ε in the limit *iff* $\exists n \forall n' \geq n : \delta(\varepsilon \mid n') \subseteq h$. The definition immediately above can be generalized. For the discovery method to discover a hypothesis, and thus project its conjecture over the entire future, in all worlds admitted by the background knowledge, say that δ discovers h *in* $[\varepsilon \mid n]$ *iff* $\exists k \forall n' \geq k \forall(\tau, n') \in [\varepsilon \mid n] : \delta(\tau \mid n') \subseteq h$.

The *convergence modulus* $cm(\delta, h, [\varepsilon \mid n])$ for the discovery method δ is the earliest possible time k such that for all later times n' the conjectures performed by the method in all the relevant possible worlds (τ, n') specified by the background assumptions $[\varepsilon \mid n]$ include a consistent conjecture entailing the hypothesis h. Thus $cm(\delta, h, [\varepsilon \mid n]) = \mu k \forall n' \geq k, \forall(\tau, n') \in [\varepsilon \mid n] : \delta(\tau \mid n') \subseteq h$.

The philosophy of science has not been sympathetic to the notion of limiting convergence. Keynes originally argued, and Kitcher [Kit93] much later follows suit, that answers with a halting property are the ones of interest because nobody lives to see the limit. Be that as it may limiting convergence allows the method to oscillate a finite, but admittedly not specifiable number of times, prior to the modulus of convergence. Thus, the method will eventually be acting on correct information it may however not know exactly when this will be the case. Even if convergence in finite time is the hallmark of convergence it does not obscure the limit as a respectable notion of convergence if the limit is what it takes for an answer. The American pragmatists Peirce and James noted, in the light of the problem of induction, that it can turn out to be impossible to say anything about the direction of science in the short run but science may nevertheless approach the truth in the long run. If one can do no better in the short run, why dismiss the long run? Surely it seems that success in the limit is better than no success at all. Martin and Osherson [MO98] argue that limiting convergence is faithful to the situation which working scientists are in because their theories are always open to revisions by new evidence until they eventually get it right albeit they might not know specifically when.

3 Hypotheses

The discovery method needs something to conjecture. In accordance with standard practice an *empirical hypothesis* is a proposition whose correctness or incorrectness only depends upon the evidence stream. Therefore hypotheses will be identified with sets of possible worlds. The set of all empirical hypotheses $\mathcal{H} = P(\omega^\omega \times \omega)$ such that $h \in \mathcal{H}$ *iff* $h = \{(\varepsilon, n) : \varepsilon \in S, n \in N\}$ where $S \subseteq \omega^\omega, N \subseteq \omega$.

Given the branching time structure, a host of different hypotheses are definable with respect to their truth-value fluctuations time as will become apparent later. For now attention is restricted to identification and limiting knowledge of laws of nature which are temporally invariant. It is thus assumed that the hypotheses of interest are *absolute time-invariant* empirical hypotheses. In order to define time invariance, define first consistency between evidence and a hypothesis: An empirical hypotheses h is *consistent* with the evidence iff $\exists(\tau, k) : (\tau, k) \in [\varepsilon \mid n] \cap h$ and $k \geq n$. From here it is possible to define absolute time invariance of an empirical hypothesis with respect to a possible world (ε, n) insofar if $(\mu, k) \in [\varepsilon \mid n] \cap h$ then $\exists \tau \in (\varepsilon \mid n) \forall l \in \omega : (\tau, k' + l) \in h$,where $k' = \max\{n, k\}$. The set $\mathcal{E}$ denotes the set of all absolute time invariant empirical hypotheses, $\mathcal{E} = \{h \in \mathcal{H} \mid h$ is absolute time invariant$\}$.

The following lemma reveals the relationship between consistency and absolute time invariance:

LEMMA 2. *Consistency and Time Invariance*

If h is absolute time-invariant w. r. t. (ε, n) and if h is consistent with the current evidence, then it is possible that h remains consistent with the evidence in all future.

It remains to define what it means for a hypothesis to be *correct*. Say, for starters, that an empirical hypothesis h which is not absolute time invariant is *epistemically correct in world* (ε, n) *iff* $[\varepsilon \mid n] \cap h \neq \emptyset$. Epistemic correctness is immensely weak—weaker than consistency with the evidence. What the relation says is that the hypothesis was at least consistent with the evidence at some point in the past.

Say next that an empirical hypothesis h is *true* in world (ε, n) iff

$$(\varepsilon, n) \in h \text{ and } \forall l \in \omega : (\varepsilon, n + l) \in h.$$

Truth is depicted Figure 1. Truth requires identification and inclusion of the actual evidence stream ε in the hypothesis for all possible future time coordinates. In other words, the actual world has to witness the hypothesis in all future even though we remain agnostic as to whether it should also witness it into the indefinite past or to the big bang.

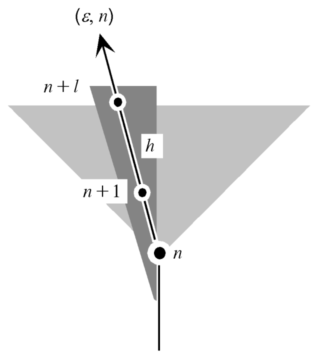

Figure 1. Truth of an absolute time invariant empirical hypotheses in a possible world.

When attention is restricted to the set of absolute time invariant empirical hypotheses, then epistemic correctness above comes very close to what Van Fraassen has called *empirical adequacy.* The notion maintains that h is empirically adequate in (ε, n) just in case there is a non-empty intersection between the background knowledge $[\varepsilon \mid n]$ and the hypothesis h. In other words given absolute time invariance of the hypothesis, there exists an evidence stream τ, that *possibly* (given the evidence observed until "now") will continue to lie within the set of worlds equivalent to the hypothesis. This evidence stream is not required to being the actual evidence stream ε. This seems to be what Van Fraassen means by empirical adequacy and *representation:*

> To believe a theory is to believe that one of its models correctly represents the world. You can think of the models as representing the possible worlds allowed by the theory; one of these possible worlds is meant to be the real one. To believe the theory is to believe that exactly one of its models correctly represents the world (not just to some extent, but in all respects). [VF80], p. 47.

Hence say that an empirical hypothesis h is *empirically adequate* in world (ε, n) iff (1) h *is* absolute time invariant, (2) $[\varepsilon \mid n] \cap h \neq \emptyset$. Van Fraassen's notion of correctness is indeed a more lenient goal than truth because of the a-symmetrical relation between the two: *A true theory is necessarily empirically adequate but an empirically adequate theory is not necessarily true*:

PROPOSITION 3. *Epistemic Correctness, Empirical Adequacy and Truth*

A	Truth $\Downarrow$ Empirical adequacy				
B	$\Downarrow$ Epistemic Correctness	$\not\Rightarrow$	Empirical adequacy	$\not\Rightarrow$	Truth
		C		**D**	

Proof.

Claim AB: Suppose a hypothesis is true in (ε, n). Then $(\varepsilon, n) \in h$ and $\forall l \in \omega : (\varepsilon, n + l) \in h$. Hence $[\varepsilon \mid n] \cap h \neq \emptyset$. Absolute time invariance requires $\exists \tau \in (\varepsilon \mid n) \forall l \in \omega : (\tau, k' + l) \in h$ where $k = k' = \max\{n, k\}$. Since ε witnesses τ the claim is obvious.

Claim CD. Consider an absolute time invariant empirical hypothesis and $\varepsilon \neq \tau$. Then the claims are obvious.

■

4 Recommendations

So far the discovery method has been defined along with non-triviality and convergence modulus. The method takes in evidence, returns hypotheses and eventually settles for some particular hypotheses which is does not back down from. It is now time to provide some specific operating instructions revealing something about how the method's conjectures are dependent on the evidence it receives eventually contributing, hopefully, to the method's non-restrictiveness and epistemic success of conjecturing the truth.

Such operating instructions are typical in methodology; basically they are integral parts of the methodological endeavour because they define what is meant by rational scientific inquiry. By way of example Popper's methodology for instance dictates that the method should generate hypothesis and subsequently test them against evidence sequences; if the hypothesis passes muster it is corroborated while if a counterinstance is found in the evidence, then the hypothesis is refuted. Since the entertainable hypotheses are enumerated according to audacity or boldness the method should, once the current hypotheses is refuted, move its pointer to the next hypotheses in the enumeration and initiate the test cycle.

These operation instructions or recommendations are like program instructions in a computer program. They tell the method how to behave observing the incoming evidence. Since consistency is a commonly advocated operation instruction say that a discovery method δ is **consistent** *iff*

$$\forall(\tau, n') : [[\tau \mid n'] \cap \delta(\tau \mid n') \neq \emptyset]\,.$$

In other words it is required that the hypotheses conjectured by the method "agrees" with the fan, or is consistent with the background knowledge. One can easily strengthen consistency to perfect memory of the evidence: Discovery method δ has **perfect memory** *iff*

$$\textit{if } (\mu, k) \in \delta(\varepsilon \mid n) \textit{ then } [(\mu \mid n = \varepsilon \mid n) \textit{ and } \forall l \leq k : (\mu, l) \in \delta(\varepsilon \mid n)]$$

In [KSH03] this property was called *data-retentiveness*. The idea is that insofar some world is in the discovery method's conjecture, then this world has the same handle as the actual world and the method "remembers" this part of the world shown to it up to state n. It remains to be demonstrated that perfect memory indeed is a strengthening of consistency: Since $\delta(\tau \mid n) \neq \emptyset$ there exists a world $(\mu, m) \in \delta(\tau \mid n)$.Perfect memory implies the *handle equality* $\mu \mid n = \tau \mid n$ and therefore $[\mu \mid n] = [\tau \mid n]$.Consequently $[\tau \mid n] \cap \delta(\tau \mid n) = [\mu \mid n] \cap \delta(\tau \mid n)$ but $(\mu, m) \in [\mu \mid n] \cap \delta(\tau \mid n)$ therefore $[\mu \mid n] \cap \delta(\tau \mid n) \neq \emptyset$ which is the definition of a consistent discovery method.

The next criterion is sort of the inverse of perfect memory. Say that a discovery method δ is **consistently expectant** *iff*

$$\textit{if } (\mu, k) \in \delta(\varepsilon \mid n) \textit{ then } [k \geq n \textit{ and } (\mu \mid n = \varepsilon \mid n)]\,.$$

Expectation is a consistency criterion but actually also a criterion related to *forcing*—the strategy of screening off irrelevant possibilities [Hen06]. It first of all requires that all the possible evidence streams on which it conjectures from k have the same handle as the handle seen. This is the consistency part of the requirement. The forcing part really shows itself later when the recommendation is paired up with a knowledge definition because then consistent expectation requires that the method only produces conjectures in the future.

Perfect memory and consistent expectation are inconsistent. On the other hand consistent expectation immediately implies consistency because of the handle-equality.

Finally consider the methodological recommendation of infallibility: δ is **infallible** *iff*

$$(\varepsilon, n) \in \delta(\varepsilon \mid n).$$

The term infallibility derives from the fact that the discovery method includes the entire world forever on the very handle of the world in question on which it is currently conjecturing. Infallibility clearly implies that the discovery method is consistent if the infallible method respects the background knowledge.

To establish the relationship between perfect memory and infallibility note initially that pertaining to perfect memory the following property holds:

PROPOSITION 4. *Perfect Memory and Joins*

If δ has perfect memory, then

$$\bigcap_{n\in\omega}\delta(\varepsilon \mid n) = \{(\varepsilon, k) \mid k \in \omega\} \textit{ or } \bigcap \delta(\varepsilon \mid n) = \emptyset.$$

There exist discovery methods with perfect memory for which the join is empty, *i. e.* the latter disjunct is empty. To see this define δ in such a way that it has both perfect memory and such that

$$\bigcap_{n\in\omega}\delta(\varepsilon \mid n) = \emptyset.$$

Let $\delta(\varepsilon \mid n) = \{(\mu^n, k) \mid k \in \omega\}$,such that $\mu^n \mid n = \varepsilon \mid n$ where $\mu^n = \varepsilon_0, \varepsilon_1, \varepsilon_2, ...\varepsilon_n, \mu^n_{n+1}, \mu^n_{n+2}, \mu^n_{n+3}...$ for which $\mu^n_{n+1} \neq \varepsilon_{n+1}$ (Figure 2). Now the claim is that

$$\bigcap_{n\in\omega}\delta(\varepsilon \mid n) = \emptyset.$$

But this follows from the fact that

$$\delta(\varepsilon \mid 0) \cap \delta(\varepsilon \mid 1) = \{(\mu^0, k) \mid k \in \omega\} \cap \{(\mu^1, k) \mid k \in \omega\} = \emptyset$$

because $\mu^0 \neq \mu^1$ since

$$\mu^0 = \varepsilon_0, \mu^0_1, \mu^0_2, \mu^0_3...$$
$$\mu^2 = \varepsilon_0, \varepsilon_{1,}\mu^1_2, \mu^1_3, \mu^1_4...$$

where

$$\mu^1_2 \neq \varepsilon.$$

The method has perfect memory but is not infallible.

Refer to the denial of the empty join as the **limiting non-triviality** property: δ *is* limiting non-trivial iff

$$\forall\varepsilon \in \omega^\omega : \bigcap_{n\in\omega}\delta(\varepsilon \mid n) \neq \emptyset.$$

Now establish the connection between the two methodological recommendations of perfect memory and infallibility:

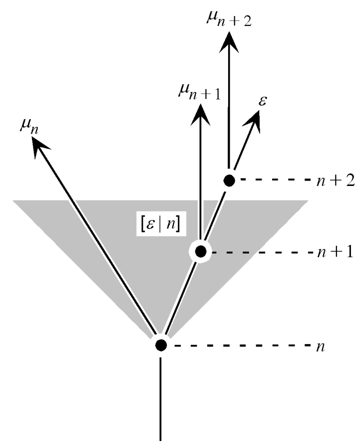

Figure 2. Perfect memory and empty join.

PROPOSITION 5. *Perfect Memory, Limiting Non-triviality and Infallibility*

If δ has perfect memory and is limiting non-trivial, then δ is infallible.

Infallibility does not imply perfect memory automatically since $(\varepsilon, n) \in \delta(\varepsilon \mid n)$ does not guarantee the conjecture's respect for the background knowledge $[\varepsilon \mid n]$. The conjecture may capture (ε, n) but also veer off at some point k earlier than n. But if the infallible method respects the background assumptions then perfect memory is clearly implied.

Finally, consistent expectation is independent of infallibility and vice versa. Consider the conjecture $\delta(\varepsilon \mid n) = \{(\tau, n+1)\}$ where $\tau \mid n = \varepsilon \mid n$ and $\tau \mid n+1 \neq \varepsilon \mid n+1$.So the discovery method has consistent expectation but is not infallible. Conversely $\delta(\varepsilon \mid n) = \{(\varepsilon, n), (\varepsilon, n-1)\}$ is infallible but not consistently expectant.

One may summarize the relations between the methodological recommendations for empirical discovery in table 1:

• (⇒) denotes the relation that one methodological norm implies another, while

• (⇏) indicates that there is no such implicational relationship,

• (⊥) denotes the fact that two recommendations are contradictory,

• (∗) signifies the use of the limiting non-triviality property.

	Infallibility	Consistent expectation	Perfect memory	Consistency
Infallibility	—	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$
Consistent expectation	$\not\Rightarrow$	—	$\perp$	$\Rightarrow$
Perfect memory	$* \Rightarrow$	$\perp$	—	$\Rightarrow$
Consistency	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	—

Table 1. Relating the Methodological Recommendations of Discovery

5 Operators

Agents may have knowledge. Based on the definiton of a discovery method, truth and the restriction to absolute time invariant empirical hypotheses one may construct a notion of knowledge for inquiring agents. The agent applying the method knows a hypothesis just in case the hypothesis is true at n, and for each later time n', in all possible worlds restricted to the background knowledge, the method produces a consistent conjecture entailing the hypothesis and the method is *infallible* by definition:

(ε, n) validates $K_\delta h$ *iff*
1. $[\varepsilon \mid n] \cap h \neq \emptyset$,
2. *h is absolute time invariant*,
3. $\forall n' \geq n, \forall(\tau, n') \in [\varepsilon \mid n]$:
   (a) $\delta(\tau \mid n') \subseteq h$,
   (b) $(\tau, n') \in \delta(\tau \mid n')$.

The definition of knowledge implies entailment of the truth, and thus empirical adequacy of h by the evidence and the background knowledge which indeed is a strong definition of knowledge. Since ε witnesses τ it is readily verified that knowledge based on discovery implies $(\varepsilon, n) \in h$ and $\forall l \in \omega : (\varepsilon, n + l) \in h$ by conditions 1 through 3 of above and hence truth of h in world (ε, n).

Modal operator theory also facillitates the definition of althetic modalities. Say that a hypothesis is **universally necessary** if and only it is correct in all possible worlds simpliciter:

$$(\varepsilon, n) \text{ validates } \Box h \text{ iff } \forall(\tau, m) \in \mathcal{W} : (\tau, m) \text{ validates } h.$$

Next, a hypothesis is said to be **empirically necessary** if and only if it is correct in all the possible worlds admitted by the background knowledge specified by the world fan:

$$(\varepsilon, n) \text{ validates } \boxminus h \text{ iff } \forall(\tau, m) \in [\varepsilon \mid n] : (\tau, m) \text{ validates } h.$$

Before moving on to introducing the standard tense operators it should be observed that one can formulate another necessity operator which quantifies over state coordinates rather than possible worlds. A hypothesis is **temporally necessary** if it is correct for all possible state coordinates:

$$(\varepsilon, n) \text{ validates } \boxdot h \text{ iff } \forall k \in \omega : (\varepsilon, k) \text{ validates } h.$$

The classical tense-logical operators can now be stated in modal operator theory:

1. (ε, n) validates Fh *iff* $\exists k > n : (\varepsilon, k)$ validates h.
2. (ε, n) validates Gh *iff* $\forall k > n : (\varepsilon, k)$ validates h.
3. (ε, n) validates Ph *iff* $\exists k < n : (\varepsilon, k)$ validates h.
4. (ε, n) validates Hh *iff* $\forall k < n : (\varepsilon, k)$ validates h.

6 Multi-Modal Logic

The above set-theoretical characterization of inquiry lends itself to a multi-modal logic. The modal language $\mathcal{L}$ used is that of the regular propositional logic augmented with the new tense, alethic and epistemic modalities.

6.1 Proof Structure

There are no complete axiomatizations of the convergent logics to come but the following transformation rules are assumed along with the rules of propositional logic:

1. *Uniform substitution* (SB)*. If* $A \in \mathcal{L}$*, then* $sA \in \mathcal{L}$ *insofar* sA *is the result of uniform substitution of formulas in* A.
2. *Modus Ponens* (MP)*. If* $(A \Rightarrow B) \in \mathcal{L}$ *and* $A \in \mathcal{L}$*, then* $B \in \mathcal{L}$.
3. *The Rule of Necessitation* (N)*:* $\frac{\models A}{\models KA}$.

6.2 Modal Semantics

Usually a model is a triple consisting of a set of possible worlds, an accessibility relation and a denotation function as in a Kripke-structure [Kri63]. The current model is very different from the traditional one in the sense that reference to an explicit accessibility relation is dropped but a scientific inquiry method added.

A model $M = <\mathcal{W}, \varphi, \delta>$ *consists of:*

1. *A non-empty set of* possible worlds $\mathcal{W}$,

2. A denotation function φ :*Proposition Letters* $\longrightarrow P(\mathcal{W})$,*i. e.*, $\varphi(a) \subseteq \mathcal{W}$.

3. An inquiry method $\delta : \omega^{<\omega} \longrightarrow P(\mathcal{W})$,

In other words, the denotation function φ assigns subsets of $\mathcal{W}$ to proposition letters a. Proposition letters denote hypotheses and, by recursion, all $\mathcal{L}$-*wffs* will denote hypotheses.

6.3 Truth-conditions

Given the model $\mathbb{M}$ state the truth-conditions for both Boolean statements and for the statements governed by the epistemic, temporal and alethic operators. The Boolean conditions follow the standard recursive recipe like:

1. $\varphi_{\mathbb{M},(\varepsilon,n)}(a) = 1\ iff\ [\varepsilon \mid n] \cap \varphi(a) \neq \emptyset$ *and* $\varphi(a) \in \mathcal{E}$.

$for\ all\ propositional\ letters\ a, b, c...\ ,.$

2. $\varphi_{\mathbb{M},(\varepsilon,n)}(\neg A) = 1\ iff\ \varphi_{\mathbb{M},(\varepsilon,n)}(A) = 0$,

3. $\varphi_{\mathbb{M},(\varepsilon,n)}(A \wedge B) = 1\ iff$ *both* $\varphi_{\mathbb{M},(\varepsilon,n)}(A) = 1$ *and* $\varphi_{\mathbb{M},(\varepsilon,n)}(B) = 1$;

otherwise $\varphi_{\mathbb{M},(\varepsilon,n)}(A \wedge B) = 0$.

and similarly for the remaning connectives.

Next, define semantic correctness in the following way. Let a be a propositional variable. Then $(\varepsilon, n) \models_{\mathbb{M}} a\ iff\ \varphi(a) = 1$. The semantical meaning of a formula A is defined as $[A]_{\mathbb{M}} = \{(\varepsilon, n) \mid (\varepsilon, n) \models_{\mathbb{M}} A\}$. Finally, *a formula A is* logically valid, *ie.* $\models_{\mathbb{M}} A\ iff\ \forall(\varepsilon, n) : (\varepsilon, n) \models_{\mathbb{M}} A$ such that a tautological statement is one of the following kind $\models_{\mathbb{M}} A\ iff\ [A]_{\mathbb{M}} = \mathcal{W}$.

The truth-conditions for knowledge based on discovery follows may be stated as:

$\varphi_{\mathbb{M},(\varepsilon,n)}(K^R_\delta A) = 1$ *iff*

1. $[\varepsilon \mid n] \cap [A]_{\mathbb{M}} \neq \emptyset$,
2. $\forall n' \geq n, \forall(\tau, n') \in [\varepsilon \mid n]$:
   (a) $\delta(\tau \mid n') \subseteq [A]_{\mathbb{M}}$,
   (b) $(\tau, n') \in \delta(\tau \mid n')$.

Truth-conditions for the tense and alethic operators are given by:

1. $\varphi_{\mathbb{M},(\varepsilon,n)}[FA] = 1$ *iff* $\exists n' > n : (\varepsilon, n') \models_{\mathbb{M}} A$.
2. $\varphi_{\mathbb{M},(\varepsilon,n)}[GA] = 1$ *iff* $\forall n' > n : (\varepsilon, n') \models_{\mathbb{M}} A$.
3. $\varphi_{\mathbb{M},(\varepsilon,n)}[PA] = 1$ *iff* $\exists n' < n : (\varepsilon, n') \models_{\mathbb{M}} A$.
4. $\varphi_{\mathbb{M},(\varepsilon,n)}[HA] = 1$ *iff* $\forall n' < n : (\varepsilon, n') \models_{\mathbb{M}} A$.
5. $\varphi_{\mathbb{M},(\varepsilon,n)}[\Box A] = 1$ *iff* $\forall(\tau, m) \in \mathcal{W} : (\tau, m) \models_{\mathbb{M}} A$.
6. $\varphi_{\mathbb{M},(\varepsilon,n)}[\boxdot A] = 1$ *iff* $\forall k \in \omega : (\varepsilon, k) \models_{\mathbb{M}} A$.
7. $\varphi_{\mathbb{M},(\varepsilon,n)}[\boxminus A] = 1$ *iff* $\forall(\tau, m) \in [\varepsilon \mid n] : (\tau, m) \models_{\mathbb{M}} A$.

7 Convergent Knowledge and S4

It is now possible to prove the first theorem of real interest. Knowledge as limiting convergence validates **S4** under the assumption that the discovery method is consistently expectant:[2]

PROPOSITION 6. *Realist Knowledge and* ***S4***
Knowledge validates ***S4*** *if the method is consistently expectant.*

8 KK-Thesis

By the proposition it follows that the KK-thesis requires use of a methodological recommendation for its validation. It is interesting to observe that none of the other methodogical recommendations, not even infallibility ensures validation of KK-thesis. That's part of being an *active agent*: The fact that the agent has to play an active role in the validation and maintenance of epistemic strengh [Hen03], [Hen06], [HS06]. Only consistent expectation does the trick. Suppose by way of example that the discovery method is infallible and suppose $(\varepsilon, n) \in [K_\delta A]$. Then

(1) $\forall n' \geq n, \forall(\tau, n') \in [\varepsilon \mid n] : \delta(\tau \mid n') \subseteq [A]$,
(2) $(\tau, n') \in \delta(\tau \mid n')$.

Then one is to show that

(3) $\forall n' \geq n, \forall(\tau, n') \in [\varepsilon \mid n] : \delta(\tau \mid n') \subseteq [K_\delta A]$,
(4) $(\tau, n') \in \delta(\tau \mid n')$.

Again (**4**) is trivially implied. Construct a counter-example from (**1**) and (**2**) to (**3**) in the following way: Given (**3**) $\delta(\tau \mid n') \subseteq [K_\delta A]$, let $(\theta, m) \in \delta(\tau \mid n')$ for which $\theta \mid n' = \tau \mid n'$ such that $(\theta, m) \in [K_\delta A]$. Hence

$$\forall m' \geq m, \forall(\nu, m') \in [\theta \mid m] : \delta(\nu \mid m') \subseteq [A].$$

Define

- $[A] = [\varepsilon \mid n]$.
- $\delta(\nu \mid k) = [\nu \mid k]$.

Then given (**1**)

$$\forall n' \geq n, \forall(\tau, n') \in [\varepsilon \mid n] : \delta(\tau \mid n') = [\tau \mid n'] \subseteq [\varepsilon \mid n].$$

Now $(\theta, m) \in [\tau \mid n']$ so for $m \leq m' < n, \nu \in (\theta \mid m) \backslash (\theta \mid n)$:

$$\delta(\nu \mid m') = [\nu \mid m'] \supset [\varepsilon \mid n].$$

[2]See [Hen01]: 203–207 for the proof.

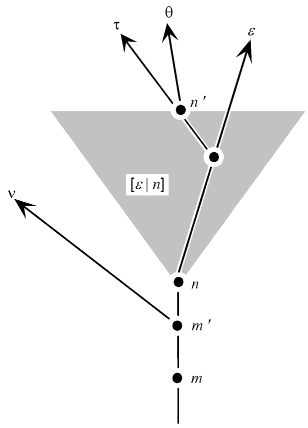

Figure 3. The counter-example to the KK-thesis.

9 Tenses

Consider the following tense-logical system:

DEFINITION 7. The Tense System $\mathbf{L}_0$

Let a and b be atomic propositions. Then $\mathbf{L}_0$ *consists of the following characteristic axioms:*

1. $G(a \Rightarrow b) \Rightarrow (Ga \Rightarrow Gb)$.
2. $H(a \Rightarrow b) \Rightarrow (Ha \Rightarrow Hb)$.
3. $a \Rightarrow GPa$.
4. $a \Rightarrow HFa$.

Then one can prove the the following proposition:

PROPOSITION 8. $\boldsymbol{L}_0$ *and Truth*

If the correctness relation is truth, then the tense operators validate $\boldsymbol{L}_0$.

Proof.

1. Consider the antecedent claims. Then $(\varepsilon, n) \vDash G(a \Rightarrow b)$ just in case $\forall k > n : (\varepsilon, k) \vDash a \Rightarrow b$and $(\varepsilon, n) \vDash Ga$ if and only if $\forall k > n : (\varepsilon, k) \vDash a$.The consequent maintains that $(\varepsilon, n) \vDash Gb$ just in case $\forall k > n : (\varepsilon, n) \vDash b$.But (iii) follows immediately by (i) and (ii) by Modus Ponens.

2. Consider the antecedent claims again. Then $(\varepsilon, n) \vDash H(a \Rightarrow b)$ just in case $\forall k < n : (\varepsilon, k) \vDash a \Rightarrow b$and $(\varepsilon, n) \vDash Ga$ if and only if $\forall k < n : (\varepsilon, k) \vDash a$The consequent maintains that $(\varepsilon, n) \vDash Gb$ just in case $\forall k < n : (\varepsilon, k) \vDash b$. But (iii) follows immediately again by (i) and (ii) by Modus Ponens.

3. Now $(\varepsilon, n) \vDash a$ if and only if $(\varepsilon, n) \in \varphi(a) \wedge \forall l \in \omega : (\varepsilon, n + l) \in \varphi(a)$.while the consequent maintains that $(\varepsilon, n) \vDash GPa$ just in case $\forall k >$

$n\exists l < k : (\varepsilon, n) \in \varphi(a) \wedge (\varepsilon, n+k) \vDash a$. Let $l = n$. Finally $(\varepsilon, n) \vDash a$ if and only if $\forall l \in \omega : (\varepsilon, n+l) \in \varphi(a)$.The consequents says that $(\varepsilon, n) \vDash HPa$ just in case $\forall k < n \exists l > k : (\varepsilon, n+k) \vDash a$. Again let $l = n$. ■

10 Alethics, Tenses and Truth

Observe the following relationships between the three necessities and truth:

PROPOSITION 9. *Truth, Universal, Temporal and Empirical Necessity*

1. $\Box A \Rightarrow \boxdot A$.
2. $\Box A \Rightarrow \boxminus A$.
3. $\boxminus A \Rightarrow \boxdot A$.

Proof. If we prove (3) and (2) then (1) follows by cut. With respect to (3), then by definition $(\varepsilon, n) \vDash \boxminus A\ iff\ \forall(\tau, m) \in [\varepsilon \mid n] : (\tau, m) \vDash A$ implies $\forall k \in \omega : (\varepsilon, k) \vDash A$ because $(\varepsilon, k) \in [\varepsilon \mid n]$. (2) follows immediately by quantifier scope so, by (3) and cut, (1) follows.

■

With respect to truth, temporal necessity and the future tenses observe the following property for which proof omitted:

PROPOSITION 10. *Truth, Temporal Necessity and the Future*

1. $\boxdot A \Rightarrow GA$.
2. $\boxdot A \Rightarrow FA$.

Similarly to proposition 10 state the relations between truth, temporal necessity and the past where proof omitted.

PROPOSITION 11. *Truth, Temporal Necessity and the Past*

1. $H \boxdot A \Rightarrow A$.
2. $P \boxdot A \Rightarrow A$.

Proposition 10 allows for the following immediate corollary for which a proof is omitted:

COROLLARY 12. *Truth, Universal, Temporal Necessity and the Future:*

1. $\boxminus A \Rightarrow GA$.
2. $\boxminus A \Rightarrow FA$.
3. $\Box A \Rightarrow GA$.
4. $\Box A \Rightarrow FA$.

11 Knowledge in Time

Consider the following combined *epistemic-tense* axiom referred to as the *axiom of futuristic knowledge* (AFK):

$$K_\delta A \Rightarrow \begin{cases} 1. & FK_\delta^x A \\ 2. & GK_\delta^x A \end{cases} . \qquad (AFK)$$

The axiom maintains that if a method a method has knowledge of a time invariant empirical hypothesis h, then this knowledge extends to either some point in the future or continuously into the future as one would expect of limiting convergent knowledge:

PROPOSITION 13. *Knowledge and AFK*

$$K_\delta^R A \Rightarrow \begin{cases} \textit{1.} & FK_\delta A \\ \textit{2.} & GK_\delta A \end{cases}$$

Proof. Assume the antecedent condition of $K_\delta A \Rightarrow GK_\delta A$ as usual. Then the following conditions hold for the antecedent

1. $\forall n' \geq n, \forall(\tau, n') \in [\varepsilon \mid n]$:
   (a) $\delta(\tau \mid n') \subseteq [A]$,
   (b) $(\tau, n') \in \delta(\tau \mid n')$.

Given 1 prove that: $\forall k > n : (\varepsilon, k) \vDash [K_\delta^R A]$,in other words: $\forall k > n$:

2. $\forall k' \geq k, \forall(\tau, k') \in [\varepsilon \mid k]$:
   (a) $\delta(\tau \mid k') \subseteq [A]$,
   (b) $(\tau, k') \in \delta(\tau \mid k')$.

Now $\forall k > n$ and $\forall k' \geq k$ reduces to $\forall k > n$ since $[\varepsilon \mid k] \subset [\varepsilon \mid n]$ by the shrinking property (SP) of the background knowledge. From this the claim is immediate. If $K_\delta A \Rightarrow GK_\delta A$ holds, then $K_\delta A \Rightarrow FK_\delta A$ by quantifier scope.

■

12 Partitioning the Hypothesis Space

Time and tenses do not only relate to knowledge and agency, but also to hypotheses. Modal operator theory is flexible enough to define different hypotheses with respect to their truth-value fluctuations over time.

12.1 Stable Hypotheses

Stable hypotheses have a partially a Kantian motivation. Kant's logical table of judgements as an *a priori* collection of all possible judgement forms is organized under four headings including one called modality under which one again finds the sub-form of *apodeictic* judgements. An apodeictic hypothesis expresses logical necessity, *i. e.* insofar the hypothesis is true it is true at all possible times (Figure 4).

DEFINITION 14. *Stable Hypotheses*

h is a stable *hypothesis iff*

1. $h \subseteq \mathcal{W}$,
2. $(\varepsilon, n) \in h \Rightarrow \forall m \in \omega : (\varepsilon, m) \in h$.

Let $\varphi(p) = h$. Then realize that by definition of temporal necessity, a stable hypothesis may also be expressed model theoretically as

$$\mathbb{M} \vDash p \Rightarrow \boxdot p$$

since a hypothesis is temporally necessary iff $\forall k \in \omega : (\varepsilon, k) \vDash h$ which is equivalent to the definition of a stable hypothesis.

The following corollary establishes the obvious connection between truth and stable hypotheses.

COROLLARY 15. *Stable Hypotheses and Truth*

If h is a stable hypothesis true in (ε, n) then $\forall m \in \omega : h$ is true in (ε, m).

Given the basic definition of simple empirical hypotheses one may provide the following characterization of stable hypothesis:

$$\boxed{h \text{ is a stable hypothesis} \Rightarrow h = a \times \omega, a \in P(\omega^{\omega}).}$$

By way of example the fan $[\varepsilon \mid n]$ is a stable hypothesis, $[\varepsilon \mid n] \cup [\tau \mid m]$ is a stable hypothesis; in general every fan is a stable hypothesis.

12.2 Eventually Stable Hypotheses

Peirce held the view that science and its theories, *i. e.* hypotheses, may converge to the truth in the limit. This means that the hypotheses of science are free to oscillate in truth-value all they care as long as they eventually stabilize to the truth and stick with it forever after no matter how the world will turn (Figure 4).

DEFINITION 16. *Eventually Stable Hypotheses*

h is an eventually stable *hypothesisiff*

1. $h \subseteq \mathcal{W}$,
2. $(\varepsilon, n) \in h \Rightarrow \exists k \forall m \geq k \in \omega : (\varepsilon, m) \in h$.

Given the tense-logical operators characterize the eventually stable hypotheses accordingly:

$$\mathbb{M} \vDash p \Rightarrow \neg \boxdot \neg Gp.$$

A special case of an eventually stable hypothesis is one in which $h \subseteq \mathcal{W}$ such that $(\varepsilon, n) \in h \Rightarrow \forall m \geq n \in \omega : (\varepsilon, m) \in h$. Hence there exists a finite initial time interval in which the hypothesis flatlines to falsity after which it changes truth-value to truth and stays with it indefinitely onward. There is a "constructivistic" trait in this definition because once h is true at a certain time it will continue to be true on the data stream. The characterization of the eventually stable hypotheses amounts to the following:

$$\boxed{h \text{ is an eventually stable hypothesis} \Rightarrow h = a \times \beta}$$

where $a \in P(\omega^\omega)$, $\beta \in P(\omega)$ such that is β co-finite, *i. e.*, $\omega \backslash \beta$ is finite.

Eventually stable hypotheses also have a finite or n-mind-change learning significance [Kel96]. Mind-change learning in formal learning theory allows the method to change its mind a finite number of times as long as it converges to the truth of the hypothesis by the time the number of mind-changes is up. Observe that the eventually stable hypotheses are allowed to fluctuate in truth-value any finite number of times as long as they n-stabilize, so eventually stable hypotheses characterize the set of hypotheses that a n-mind-change learner in the *worst* case can be asked to identify—and can succeed on.

12.3 Oscillating Hypotheses

Reading Quine radically enough suggests that not even the most basic hypotheses in the web of belief are eternally true. In principle every hypotheses can continuously change truth-value; the fact that they do not is secured by our methodological prescriptions like conservatism, simplicity etc. and due to our apprehensions toward eternal belief revision. In this worst case define oscillating hypotheses accordingly:

DEFINITION 17. *Oscillating Hypotheses*
h is an oscillating *hypothesis iff*
$(\varepsilon, k) \in h \Rightarrow \forall n \exists n_1 \exists n_2 : n_1 \geq n \wedge n_2 \geq n \wedge (\varepsilon, n_1) \notin h \wedge (\varepsilon, n_2) \in h.$

The tense-logical description of oscillating hypotheses amounts to

$$\mathbb{M} \vDash p \Rightarrow \boxdot(Fp \wedge F\neg p).$$

Note that the definition of oscillating hypotheses implies

$$\exists f_1 : \omega \longrightarrow \omega,$$
$$\exists f_2 : \omega \longrightarrow \omega,$$

for which both f_1 and f_2 are increasing functions such that

$$\{(\varepsilon, f_1(n) \mid n \in \omega\} \subseteq h,$$
$$\{(\varepsilon, f_2(n) \mid n \in \omega\} \cap h = \emptyset.$$

Here is an example of an oscillating hypothesis:

$$h = \{(\varepsilon, f(n) \mid n \in \omega, f \text{ is not recursive and } f \text{ on } \omega \text{ is infinite}\}.$$

Hence f is choice-function which picks out a non-recursive succession of natural numbers. The 'tense' -demon of induction is hence allowed to pick a function of arbitrarily high complexity, i.e., the function may be taken from any complexity hierarchy above the arithmetical. Another similar example would be

$$h = \{(\varepsilon, f(n) \mid n \in \omega, f(n+1) > 2 \cdot f(n)\}.$$

12.4 Initially Stable Hypotheses

Kuhn was of the opinion that the truth-value of a scientific hypothesis is dependent upon the current paradigm entertained. A hypotheses true in one paradigm may come out false in another paradigm and yet turn true again in another succeeding the former (Figure 4).

DEFINITION 18. *Initially Stable Hypotheses*
h is an initially stable *hypothesis iff*
1. $h \subseteq \mathcal{W}$,
2. $(\varepsilon, n) \in h \Rightarrow \exists k \forall m \leq k \in \omega : (\varepsilon, m) \in h$.

In other words, the initially stable hypotheses have the following tense-logical counterpart:

$$\mathbb{M} \vDash p \Rightarrow \neg \boxdot \neg Hp.$$

Give the following characterization of initially stable hypotheses:

$$\boxed{h \text{ is an initially stable hypothesis} \Rightarrow h = \bigcup_{\varepsilon \in a} \{\varepsilon\} \times \omega_\varepsilon}$$

where $\omega_\varepsilon = \{k \mid k \leq m(\varepsilon)\}$ and $a \in P(\omega^\omega)$.

12.5 Absurd Hypotheses

Finally there are hypotheses in science that can scare the scientists more than anything. These are the ones that either look like they are eternally true, or look like they are forever oscillate in truth-value. But the truth to the matter is that they will eventually drop to falsity and stick with it. Hence either the scientist may come to think that the hypothesis is true, or he will settle for the conclusion that it forever jumps between truth and falsity; *either way he fails*. Refer to these beasts as the *absurd* hypotheses:

DEFINITION 19. Absurd Hypotheses

h is an absurd *hypothesis iff*
1. $h \subseteq \mathcal{W}$,
2. $(\varepsilon, n) \in h \Rightarrow \exists k \forall m \geq k \in \omega : (\varepsilon, m) \notin h$.

By definition an absurd hypothesis is not absolute time-invariant. Alternatively they correspond to the following tense-logical description

$$\mathbb{M} \vDash p \Rightarrow FG\neg p.$$

The absurd hypotheses are the only ones which do not contain an infinite data stream. However, any initial segment of such a hypothesis can be viewed as compatible with both a stable and oscillating hypotheses. Adopting the the third person perspective on inquiry we know that absurd hypotheses eventually will drop to falsity and stick with it forever after but

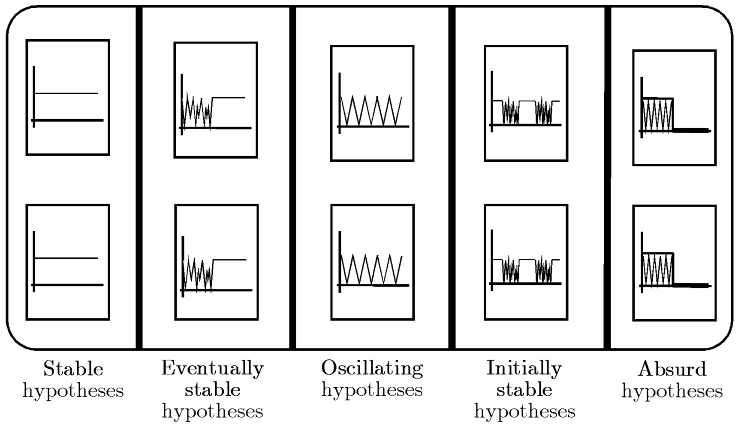

Figure 4. Partitioning the hypothesis space

the initial segment on which it does not drop can be arbitrarily long. With respect to absurd hypotheses the method can never be sure to converge even though the method at any given time on the initial segment of the apparently stabilized absurd hypothesis may be fooled into doing so, and even worse, supposing that the method is allowed an infinite number of mind-changes it is still destined to failure.

13 Learning Hypotheses

What classes of hypotheses may a limiting scientific discovery method identify? Observe first that eventually stable and initially stable hypotheses are dual in the sense that eventually stable hypotheses are almost always true while in the worst case initially stable hypotheses are almost always false. In general the following relationships hold between the five kinds of hypotheses:

LEMMA 20. *Hypothesis Relations*

1. *stable hypotheses* $\Rightarrow$ *eventually stable hypotheses (or* $\boxdot h \Rightarrow \neg \boxdot \neg Gp$*).*
2. *stable hypotheses* $\Rightarrow$ *initially stable hypotheses (or* $\boxdot h \Rightarrow \neg \boxdot \neg Hp$*).*
3. *oscillating hypotheses* $\cap$ *initially stable hypotheses* $\neq \emptyset$.
4. *absurd hypotheses* $\cap$ *initially stable hypotheses* $\neq \emptyset$.
6. *stable hypotheses* $\cap$ *oscillating hypotheses* $= \emptyset$.
7. *eventually stable hypotheses* $\cap$ *oscillating hypotheses* $= \emptyset$.

13.1 Unidentifiability Results

Some hypotheses create grave problems for learners and their methodological recommendations:

PROPOSITION 21. *Unidentifiability of Oscillating Hypotheses*
An oscillating hypothesis is not limiting identifiable by an infallible discovery method δ.

Proof. Assume that h is an oscillating hypothesis and assume $(\varepsilon, n) \in h$. Then there exists a strictly increasing function $f_2 : \omega \longrightarrow \omega$ such that $\{(\varepsilon, f_2(n) \mid n \in \omega\} \cap h \neq \emptyset$.But note that given limiting identification and infallibility of the discovery method one obtains

$$\left.\begin{array}{l} \exists k \forall m \geq k : \delta(\varepsilon \mid m) \subseteq h \\ (\varepsilon, m) \in \delta(\varepsilon \mid m) \end{array}\right\} \Rightarrow \exists k \forall m \geq k : (\varepsilon, m) \in h$$

which contradicts the definition of the oscillating hypothesis $\{(\varepsilon, f_2(n) \mid n \in \omega\} \cap h \neq \emptyset$.

■

COROLLARY 22. *Unidentifiability of Initially Stable Hypotheses*
There exists initially stable hypotheses which are not limiting identifiable by an infallible discovery method δ.

Proof. Choose a hypothesis which is both initially stable and oscillating. Such a hypothesis exists by lemma 20. Then the corollary is immediate by proposition 21.

■

COROLLARY 23. *Unidentifiability of Absurd hypotheses*
An absurd hypothesis is not limiting identifiable by an infallible discovery method δ.

Proof. The proof is a simplified version of the above argument. Hence suppose that δ infallible and identifies an absurd h in the limit, *i. e*:

1. $(\varepsilon, k) \in \delta(\varepsilon \mid k)$,
2. $\exists k \forall m \geq k : \delta(\varepsilon \mid k) \subseteq h$.

Then $\forall m \geq k : (\varepsilon, m) \in \delta(\varepsilon \mid m) \subseteq h$,hence $\forall m \geq k : (\varepsilon, m) \in h$, which again contradicts the definition of an absurd hypothesis.

■

13.2 Identifiability Results

The above propositions are based on the assumption that the discovery method is infallible. However if we choose to drop infallibility and settle for perfect memory it is possible to construct discovery methods which may identify some oscillating, initially stable and absurd hypotheses:

PROPOSITION 24. *Consistency, Oscillating, Initially Stable and Absurd Hypotheses*

There exist oscillating, initially stable and absurd hypotheses which are limiting identifiable by a discovery method δ with perfect memory.

Proof. Taking the oscillating hypotheses first let h be an oscillating hypothesis such that $h = \{(\varepsilon, f(n)) \mid n \in \omega\}$ such that $f : \omega \longrightarrow \omega$ is increasing and the range of f is a proper subset of ω. Next define δ recursively such that it has perfect memory and discovers h in the limit: Let $k_0 = f(0)$. If $k \leq k_0$ then $\delta(\varepsilon \mid k) = \{(\varepsilon, k_0)\}$. If $k > k_0$, then

$$\delta(\varepsilon \mid k) = \begin{cases} \delta(\varepsilon \mid k-1) \cup \{(\varepsilon, k)\} & if\ \exists n : k = f(n), \\ \delta(\varepsilon \mid k-1) & otherwise. \end{cases} .$$

Clearly δ perfect memory and also generates the oscillating hypothesis by the following meet:

$$\bigcup_{k \in \omega} \delta(\varepsilon \mid k) = h.$$

Next, there exist initially stable hypotheses which are also oscillating ones, so the proof is immediate.

Finally regarding the absurd hypotheses let h be the hypothesis defined by the following meet:

$$h = \bigcup_{k>0} h_k, h_k = \{(\varepsilon_k, m) \mid 1 \leq m \leq k\}.$$

Hypothesis h is clearly absurd. Next define δ such that it has perfect memory and discovers h in the limit, *i. e.:*

1. $\delta(\varepsilon_k \mid n) = h_k$ for all $k > 0$ and $n \in \omega$,
2. $\delta(\mu \mid n) = \{(\mu, n)\}$ when $\mu \neq \varepsilon_k, n \in \omega$.

This suffices for conjecturing h.

■

The above results are not exhaustive, completeness results and characterization theorems of learnability are still missing. All the same, the above sample of propositions is interesting because they balance methodological strength in learnability against the temporal complexity of the hypotheses classes.

In [HP03] it was demonstrated how infallibility and perfect memory in general are *truth-conducive methodological prescriptions for discovery methods in search of the truth of absolute time invariant empirical hypotheses*. This extends to some of the oscillating, initially stable and absurd hypotheses. On the other hand the infallibility prescription actually stands in the way of finding the truth when it can be found by a method furnished with the different and to some extent weaker criterion of perfect memory. In order for a method to be infallible when it has perfect memory it has to assume limiting non-triviality in accordance with proposition 5.

Until now we have restricted attention to identification not knowledge as such. But given the definition of knowledge as limiting convergence for a discovery method one may finally ask what a discovery method based on limiting convergence can know rather than just identify.

PROPOSITION 25. *Incompleteness over Oscillating, Initially Stable and Absurd Hypotheses*

If δ is limiting non-trivial, then knowledge is identificationally incomplete over oscillating, initially stable and absurd hypotheses.

14 Conclusion

With the temporal indices on the hypotheses, the epistemic game changes: Some hypotheses are simply beyond identificational reach even for a very powerful and truth-conducive discovery method. This is a serious issue because then succeeding on epistemic axioms is far from trivial when paired with temporal fluctuations of hypotheses. To identify and know is then relative not only to background assumptions, the truth and methodological recommendations but also to the possibly tensed nature of the hypotheses under investigation:

> *An inquiry method or agent can fail to know a hypothesis not only due to either silly or too demanding methodological recommendations getting the method off-track but also due to the complex temporal nature of the hypotheses.*

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