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Modal Operator Theory

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Introduction

Formal treatments of epistemology and epistemological issues have come a great way in recent years. Valuable results have been produced by Bayesians, in formal learning theory (also known as computational epistemology in [Kelly 96] and epistemic logic. While the first to formal approaches concentrate on learning and knowledge acquisition issues, the latter concentrate on axiomatics and validity.

Then there is also mainstream epistemology, which seeks necessary and sufficient conditions for the possession of knowledge by presenting 'intuitive' examples and counterexamples. All these various approaches have proceeded largely in isolation from one another. This seems to be the state of epistemology today.

What we have dubbed modal operator theory (MOT) [Hendricks 01], [Hendricks & Pedersen 00a] is a paradigm obtained by mixing epistemic, tense and alethic logic with a few concepts drawn from computational epistemology. The paradigm can then be used to study the acquisition and validity of knowledge observing the apparatus and insights of the formal and mainstream epistemologies.

Now modal logic has grown into a mature field of research with a wide range of applications in both philosophy, linguistics and computer science. Nevertheless approximately 30 years ago, Dana Scott in his famous article 'Advice on Modal Logic' pointed out a problem for the entire modal logic endeavor, which still to this day largely holds true:

Here is what I consider one of the biggest mistakes of all in modal logic: concentration on a system with just one modal operator. The only way to have any philosophically significant results in deontic logic or epistemic logic is to combine these operators with: Tense operators (otherwise how can you formulate principles of

change?); the logical operators (otherwise how can you compare the relative with the absolute?); the operators like historical or physical necessity (otherwise how can you relate the agent to his environment?); and so on and so on. [Scott 70], p. 143.

The criticism is obviously quite severe both theoretically and for applications. In recent years however modal logicians in general have begun to take Scott's seminal criticism into account. In branching tense logic one has begun to mix alethic and tense logical operators [Braüner 98a], [Zanardo 96] and the valuable and influential work by notably [Fagin et al. 95] combines epistemic logic with tense logic.

Another problem for notably epistemic logic is whether or not it should be designed to supplement epistemology in general. One should think so since epistemic logic is the logic for knowledge and belief. Classical epistemic logicians, [Hintikka 62] and in particular in [Lenzen 78], are of a different opinion. Epistemic logic is a field of its own and rather remote field from traditional epistemological problems like justification.

But since it is a scientific method or agent, which is ascribed knowledge, it is indeed a natural consequence that whether the method obtains knowledge or not should be acutely sensitive to what the agent decides to do. That is what contemporary epistemic logicians like [Fagin et al. 95] believe and one of the very cornerstones of modal operator theory.

A Characterization of Inquiry

Modal operator theory initially includes a specification of various items like evidence streams, possible worlds, hypotheses, background knowledge, inquiry methods etc. We will review some of the main items below but leave out the technical details.

Inquiring agents need something to inquire about. Usually this is taken to be the world and sometimes other possible worlds. MOT requires some very explicit structure on possible worlds adopted largely from [Kelly 1996]. Accordingly, a possible world is taken to be a pair (ε, n) consisting of ω -sequence of natural numbers, ε , and a state co-ordinate, n . The set of all possible worlds is denoted \mathcal{W} .

Thus, the picture of scientific inquiry is the following. The agent starts out by observing ever-growing sequences of evidence. Eventually the agent will perform a response in terms of a conjecture (hypothesis) pertaining to the world's infinite trajectory. The amount of evidence observed up until, say, time n is called the *handle*, $\varepsilon \upharpoonright n$. From $\varepsilon \upharpoonright n$ the world may take any turn it pleases. So there is a whole *fan* of possible ways (worlds) the world could be (all evidentially consistent with the handle) one of which is the actual. This fan is also denoted $[\varepsilon \upharpoonright n]$ and referred to as the *background knowledge* (Figure 1).

Adopting background knowledge is not uncommon. Baysians adopt background knowledge while assigning priors, epistemic logicians do so too when determining the relational properties of the accessibility relation, and Nozick is a mainstream epistemologist which defines knowledge relative to close worlds in accordance

with the standard semantics of the subjunctive conditional. The agent may accordingly neglect certain possibilities even though they are logically possible. Neglecting possibilities or possible worlds is part of being a *forcing* epistemology: Whenever knowledge claims are challenged by alleged possibilities of error, the strategy is to show that the possibilities of error fail to be genuine in the relevant sense. [Hendricks 02].

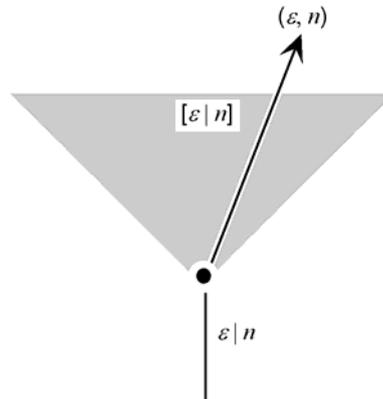


Figure 1

This conception of possible worlds suffices for imposing a tense structure—more specifically an Ockhamistic branching time tense-structure with a starting point (Figure 2) with possible worlds as chronicles.

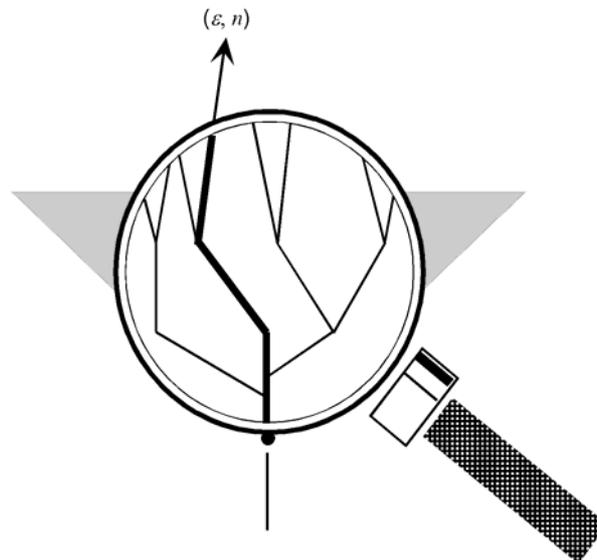


Figure 2

Hypotheses are taken to be sets of possible worlds. Truth of a hypothesis in a possible is identification of actual world forever after (Figure 3).

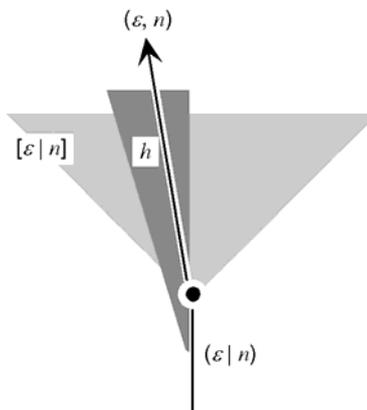


Figure 3

An inquiring agent or inquiry method, in particular a discovery method is a function from finite initial segments of evidence to hypotheses.

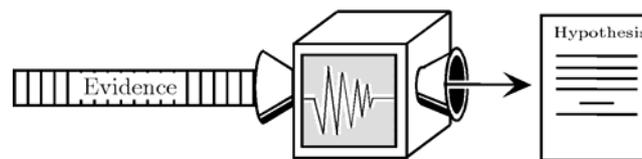


Figure 4

Now such a method may be subject to various methodological constraints or recommendations. A discovery method may for instance

- be **consistent** in the sense that its current conjecture is consistent with the evidence observed,
- have **perfect memory** in the sense that it remembers the evidence observed,
- have **consistent expectation** in the sense that it always directs its conjectures into the future in a special way.
- be **infallible** in the sense that its current conjecture is entailed by the evidence.

A variety of *limiting* concepts of knowledge are definable based on discovery methods. The idea of weakening the notion of convergence from ‘heureka’ convergence¹ to a limiting criterion is to allow for more inductive problems to come within the scope of inquiry. Limiting convergence is not a novel concept.

¹ The method will *halt* with the correct answer in each possible world admitted by the background knowledge.

Peirce held the view that it is impossible to say anything about the direction of science in short run but science may all the same asymptotically approach the truth in the long run. James distinguishes pragmatism from ‘absolutist philosophy’ by the former philosophy’s realization that science can satisfy the truth relation without necessarily being able to tell when it has succeeded. In view of the problem of induction, James correctly observed that knowledge of general laws is impossible if one is required to say when science has succeeded. Since James and Peirce, limiting convergence has become a more and more respected convergence criterion both in philosophy, computer science and in methodology. Formal learning theory uses limiting convergence for obtaining certain characterization theorems while Bayesians apply a limiting convergence criterion to facilitate ‘almost sure’ convergence theorems, etc.

Thus, say that a discovery method δ converges to hypothesis h in the limit if there exists a time n such that for each later time n' : δ conjectures h at n' . The time by which δ stabilizes to h is called the *modulus of convergence* (Figure 6). By definition the discovery method does not necessarily present an ‘heureka’ which accounts for the limiting label.

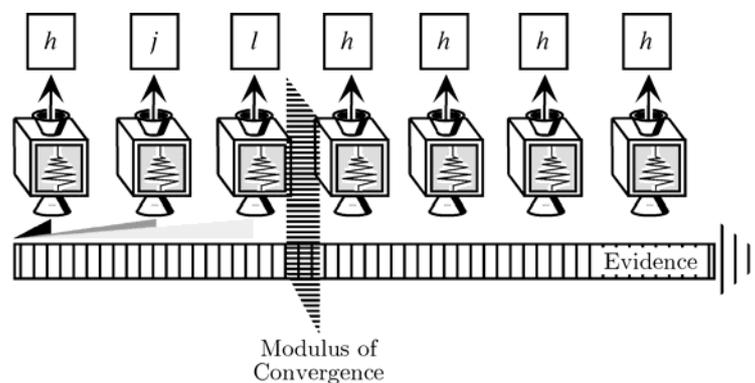


Figure 6

Based on limiting convergence one may proceed to define a concept of limiting convergent knowledge:

δ may know hypothesis h in the limit, iff there exists a possible world which validates δ 's knowledge of h , i. e.

- (a) *h is true,*
- (b) *δ conjectures h after some finite evidence sequence has been read and continues to conjecture h in all future.*

Compared to the standard JTB account of knowledge, the truth condition remains the same, conjecture is substituted for belief and the troublesome justification condition is replaced by ‘convergence in the right way.’ The right way is again specified by the method’s conjectural behaviour while observing the evidence. In

order for the method to validate various epistemic axioms (see the next section), the method may have to follow various methodological recommendations to succeed. Consequently, justification has to do with methodology, i. e. the study of the methods and recommendations with which science arrives at its posited truths.

From Characterization to Modal Logic

The set-theoretical characterization allows for formalization in a modal propositional calculus. ‘An agent knows that A ’ is formalized as a modal operator in a formal language, which is interpreted using the standard apparatus of modal logic. The language accordingly includes the Boolean operators and the epistemic operator $K_{\delta}h$ formalizes the concept of limiting convergent knowledge defined above.

The current framework is flexible enough to formulate both alethic and temporal modalities. By way of example, one may introduce a universal necessity operator in the following way:

*A possible world (ε, n) validates h iff for all possible worlds (τ, m) in W :
 (τ, m) validates h .*

Similarly one may also define the standard temporal operators F and H like:

*A possible world (ε, n) validates Fh iff there exists a $k > n$:
 (ε, k) validates h .*

*A possible world (ε, n) validates Hh iff for all $k < n$:
 (ε, k) validates h .*

How to define the two dual temporal operators G and P is obvious.

In standard epistemic logic one proceeds axiomatically. The idea is to determine the strength of the epistemic operator by investigating which epistemic axioms the epistemic operator may validate. Celebrated axioms include:

Axiom (T): $K_{\delta}h \rightarrow h$.

Axiom (K): $K_{\delta}(h \rightarrow h') \rightarrow (K_{\delta}h \rightarrow K_{\delta}h')$.

Axiom (4): $K_{\delta}h \rightarrow K_{\delta}K_{\delta}h$.

Axiom (5): $\neg K_{\delta}h \rightarrow K_{\delta}\neg K_{\delta}h$.

In proper combination these axioms make up classical modal logical systems where $T + K + 4 = S4$ while $T + K + 5 = S5$.²

One can now ask the following pair of questions:

² For a Danish introduction to modal and epistemic logic, see [Hendricks & Pedersen 02].

- ? Which epistemic axioms (and systems) can be validated by an epistemic operator based on the definition of limiting convergent knowledge?
- ? Does the validity of the various epistemic axioms (and systems) relative to the method depend upon enforcing methodological recommendations?

The first question is of course the typical question of epistemic logic, the second one is either implicitly evaded or dodged explicitly in the standard approach to epistemic logic:

Though there is basic agreement that something must be added to true belief to obtain knowledge, what precisely this ‘something’ is, remains far from being evident. Because of the vagueness of such notions as ‘having sufficient reasons for believing’, ‘being justified in believing’, it is difficult to make a decision concerning the adequacy of (5), i.e. that knowledge implies justification. [Lenzen 78]: 28.

Lenzen continues:

The search for the correct analysis of knowledge, while certainly of extreme importance and interest to epistemology, seems not significantly to affect the object of epistemic logic, the question of validity of certain epistemic-logical principles. [Lenzen 1978]

But of course epistemic logic should hook up with broader epistemological issues and one way in which the two can supplement each other is this: Epistemic-logical principles or axioms are relative to an agent or method, which may or may not validate these principles. The primary role of the methods in classical epistemic logic is to index the accessibility relation. Indices on accessibility relations will not suffice for epistemological pertinence simply because there is nothing particularly epistemic about being indices. The method should perform an active role in the validation process—i.e. possibly behave in certain ways specified by the methodological recommendations. Then it will gain epistemological pertinence and the troublesome justificational issue will come within scope of epistemic logic. Here is a theorem of MOT to prime the pump.

If knowledge is defined as limiting convergence, then knowledge validates S4 if the discovery method has consistent expectations.
[Hendricks 01]

It can be proved that:

- Axiom (T) does not require enforcing methodology for validity.
- Axiom (K) does not require enforcing methodology for validity.
- Axiom (4) (the *KK*-thesis) does require enforcing methodology for validity!

Consequently, a methodological recommendation may according to MOT be either:

Boosting in the sense that the methodological recommendation is conducive towards validating epistemic axioms and systems.

Debilitative in the sense that the methodological recommendation is an impediment towards validating epistemic axioms and systems.

Neutral if it is neither boosting nor debilitative.

The KK-thesis and the Limit

It is only with respect to Axiom (4) or the *KK*-thesis that one has to require that the method behaves in accordance with a methodological recommendation.

It has been argued since James that the *KK*-thesis and limiting convergence are incompatible:

This does not entail that Θ knows he knows the answer, since Θ may lack any reason to believe that his hypotheses have begun to converge. [Martin & Osherson 98].

The idea is this. Given the definition of limiting convergence, there exists a modulus of convergence, but the method may not be able to tell when this modulus of convergence has arised. Thus, the method may know without knowing that it knows.

How to have the cake and eat it too? In order to make sense of eating the cake a few additional concepts and distinctions must be introduced. First say that

- an epistemic or doxastic axiom is **synchronic** if the consequent obtains by the very same time the antecedent obtains,

while

- an epistemic or doxastic axiom is **diachronic** if the consequent either obtains later or would have obtained later than the antecedent even if things had been otherwise.

Usually epistemic axioms are interpreted synchronically—probably because the temporal or counterfactual dimensions have not been considered. But if the framework is strong enough to handle these other operators then there is nothing wrong with interpreting the epistemic axioms diachronically.

Additionally, say that

- a perspective on scientific inquiry is **1st person** if it is considered what an agent can solve, can do or defend considering the available means for an end given the epistemic environment he is sunk into.
- a perspective on scientific inquiry is **3rd person** if it is considered what an agent could solve, could do or defend considering the best means for an end independently of the epistemic environment he is sunk into

The distinction between 1st and 3rd is a relatively new distinction. For instance, Levi's epistemological program [Levi 97] is essentially a garden-variety of a 1st person methodology in which the main issue in the semantics for Levi is Ramsey's distinction between a *logic of truth* and a *logic of consistency*, rather than first and third person distinction. These two sets of distinctions are obviously related but not exactly identical. Levi argues against the validity of the *KK*-thesis as an axiom of an epistemic logic of truth which, somewhat simplified, is tantamount to denying that the *KK*-thesis is an axiom for a third person knowledge operator. What Levi really argues is that the *KK*-thesis is valid as a principle regulating the consistency of a rational epistemic agent while the logic of truth for epistemic agents need not be regulated by such a principle. Lewis [Lewis 96] seems to follow suit and the underlying suggestion must in the end be that if there is a universal third person logic of knowledge, such a logic is probably rather weak.

From the MOT point of view it is however possible to validate the *KK*-thesis when recognizing that the *KK*-thesis can be a diachronic axiom from the third person perspective. The method has to converge to the fact that it knows *h*, then wait around and then later converge to the fact that it knows that it knows *h* even if this is impossible from the 1st person perspective. Only *consistent expectation* allows for this kind of strategic forcing relative to the background knowledge when all the other recommendations can be shown to fail.³

Levi is an example of an epistemologist defending the validity of the *KK*-principle as a synchronic principle, and arguing in addition that the principle is not universally valid under the third person perspective. On the face of it, the current result may seem to reverse the situation: If the *KK*-thesis is valid then it is valid diachronically from the 3rd person perspective and cannot be valid synchronically from the 1st person perspective. This is however not the case. It is neither being claimed that the *KK*-thesis is impossible to validate from the 1st person perspective nor that it is impossible to validate synchronically in the 1st person. An autoepistemologist like Malcolm [Malcolm 56] is an example of an epistemologist arguing for exactly this combination. There are other models of first person knowledge operators validating *KK* in particular [Moore 95], [Fitting 83] and [Arlo Costa 98]. Actually these models yield a stronger logic than S4—they validate S5!⁴

Two things are currently being claimed:

1. The *KK*-thesis is possible to validate from the 3rd person point of view if the principle is understood diachronically and thus undermines the idea that *KK*-thesis and limiting convergence are incompatible.
2. There is a lot more to validating epistemic axioms than indices on accessibility relations: Methodological recommendations, synchronicity

³ Actually it can be proved that consistency and infallibility are neutral methodological recommendations but perfect memory is a debilitating recommendation. [Hendricks 01]

⁴ It can on the other hand be shown that the Axiom (5) and consequently S5 is not possible to validate in modal operator theory when knowledge is based on limiting convergence. When knowledge is defined as limiting convergence it contrapositively follows that if the method has not converged the method does not know. Non-convergence cannot be converted into convergence. Thus, no non-knowledge into knowledge.

and diachronicity and 1st vs. 3rd person perspectives on scientific inquiry (Figure 7). MOT tuned to epistemology provides a reply taking these parameters into account.

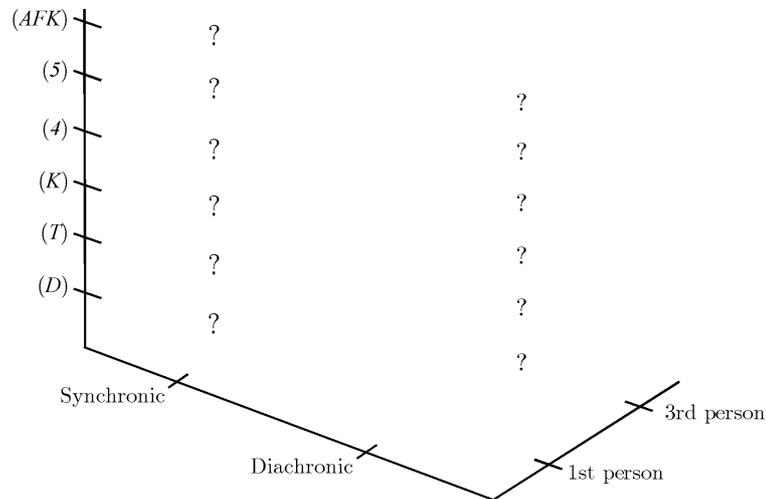


Figure 6

In conclusion, modal operator theory has at its base rather than as a derivative the idea that whatever epistemic axioms and epistemic systems are possible to validate for some epistemic operator is acutely sensitive to the methodological behavior of the agent involved.

Additionally, the paradigm provides a new unifying modal logical framework in which alethic, tense and epistemic operators can be defined and thus begins to realize what Dana Scott has long wished for.⁵

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⁵ MOT has its own homepage—The Online Companion to Modal Operator Theory: <http://www.mot.ruc.dk>.

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