Characteristic Cascades

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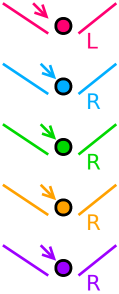
Introduction



Informational Cascade

- Each agent acts in turn, based on a *private* and a *public signal.*
- There is a *tipping point* after which previous agents' choices dominate, and a *cascade* is started.

Basic Idea

We have a set of agents $I = \{1, 2, ..., n\}$, which act in turn. Each turn goes:

1. Agent i has some beliefs based on a *private signal* (prior beliefs, issued ticket,...).
2. The agent receives a *public signal* by the actions of agents $1, ..., i-1$.
   This produces first model for agent i, M_i.
3. The agent *deliberates* by considering presented information.
   This produces second model for agent i, M_i'.
4. Based on M_i', the agent applies a *decision rule* and acts.

The outcome depends on the *interpretation rule* used in step 2, the *merge rule* in step 3 and the *decision rule* in step 4

Background: Plausibility Models

Models:

$$M = (S, \sim_i, \leq_i, \|\cdot\|)_{i \in I}$$

1. S is a finite set of *states*.
2. $\sim_i$ is an equivalence relation on S, for each $i \in I$. Let $i(s) = \{t : s \sim_i t\}$.
3. $\leq_i$ is a preorder locally connected in $i(s)$, for all $s \in S$.
4. $\|\cdot\|$ is a valuation map, $\|\cdot\| : \Phi \longrightarrow \mathscr{P}(S)$.

Language:
$\Phi = \{p\}$ and set $L := p$, $R := \neg p$. For $i \in I$, wff are:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid B_i\varphi \mid K_i\varphi$$

Semantics:
$M, s \models p$ iff $s \in \|p\|$
$M, s \models B_i\varphi$ iff $\forall t \in Min_{\leq_i}(i(s)) : M, t \models \varphi$ wh.
$Min_{\leq_i} = \{t \in i(s) : t \leq_i s', \text{ for all } s' \in i(s)\}$
$M, s \models K_i\varphi$ iff $\forall t \in i(s) : M, t \models \varphi$
Interpretation:

Minimal means Most Plausible ("*less means more*"): $s <_i t$ "$=$" $s \longleftarrow_i t$.

Signals and Actions

Actions:

$\alpha = \{L, R\}$, $\alpha_i \in \alpha$.

Denote by σ the full string of chosen actions: $\sigma = \alpha_1, \alpha_2, ..., \alpha_n$

Private Signal:

$\sigma_i \in \alpha \cup \{\lambda\}$

σ_i forms the agent's initial, "private" beliefs about L/R:

If $\sigma_i = L/R$, then $M_i, s_0 \models B_i(L/R)$

Public signal:

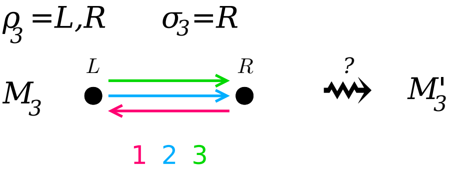
Initial segment of σ of length $i-1$:

$$\rho_i = \alpha_1, ..., \alpha_{i-1}.$$

$\rho_1 = \lambda$, the empty string.

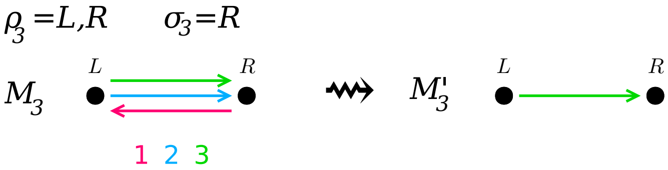
ρ_i forms the agent's higher order beliefs depending on interpretation of actions.

Deliberation?



- We just want the agents to count heads
  - Order of previous agents does not matter
  - This means conservative ($\uparrow_i L, \uparrow_i R$) / radical upgrade ($\Uparrow_i L, \Uparrow_i R$) does not work ($i$'s beliefs will just oscillate)
- No Priority
  - So Priority Merge ($R_{j \otimes i}$) and Lexicographic Merge ($R_{j/i}$) is out: i will just believe j.
- Has this been treated in the literature? Can we use things from judgment aggregation?
- Here, we construct a merge procedure: *Majority Merge*

Majority Merge



Majority merge is a belief upgrade method by which a single agent can change his beliefs regarding p based on the perceived beliefs of a group of other agents.

The idea behind the merge is 'head counting': agent i compares how many agents j in group G hold that $s <_j t$ for some two states s and t, and **adopts the opinion of the majority**.

Majority Merge

For agent a and group G and L/R:

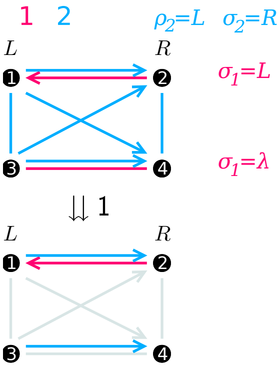
1. For each i, find $\prec_i = <_i \cap \bigcap_{i \in G} \sim_i$.
2. Find $\prec = \biguplus_{i \in G} \prec_i$, the multiset sum of $\prec_i$, $i \in I$.
3. Remove symmetries: if there are pairs (s,t) and (t,s) in $\biguplus_{i \in G} \prec_i$ then remove one of each until only one direction remains. Denote the resulting set $\bar{\prec}$.
4. Set $\leq'_a := \bar{\prec} \cup (\sim_a \cap \|L\| \times \|L\|) \cup (\sim_a \cap \|R\| \times \|R\|)$, closed under transitivity.

If the states are arranged properly this amounts to
counting horizontal arrows (1-3) and **restoring the information cell** (4)

Majority Merge: Example 1

Step 1. For each i, find $\prec_i = <_i \cap \bigcap_{i \in G} \sim_i$.

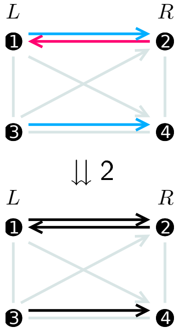
$\prec_1 = \{(1,2)\}$
$\prec_2 = \{(2,1),(4,3)\}$



Majority Merge: Example 1

Step 2. Find $\prec = \biguplus_{i \in G} \prec_i$, the multiset sum of $\prec_i$, $i \in I$.

$\prec = \biguplus_{i \in G} \prec_i = \prec_1 \uplus \prec_2 =$
$\{(1,2),(2,1),(4,3)\}$

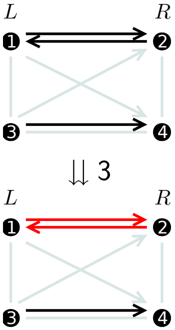


Majority Merge: Example 1

Step 3. Remove symmetries: if there are pairs (s,t) and (t,s) in $\biguplus_{i\in G} \prec_i$ then remove one of each until only one direction remains. Denote the resulting set $\bar{\prec}$.

$$\prec_1 \uplus \prec_2 = \{{\color{red}(1,2)},{\color{red}(2,1)},(4,3)\}$$

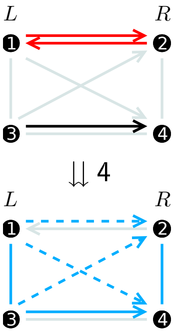
$$\bar{\prec} = \{(4,3)\}$$



Majority Merge: Example 1

Step 4. Set $\leq'_a := \bar{\prec} \cup (\sim_a \cap \|L\| \times \|L\|) \cup (\sim_a \cap \|R\| \times \|R\|)$, closed under transitivity.

$\leq'_a = \{(4,2)\}$
$\cup \{(1,1),(1,3),(3,1),(3,3)\}$
$\cup \{(2,2),(2,4),(4,2),(4,4)\}$
$\cup \{(2,1),(2,3),(4,1)\}$



So, being given $\rho_2 = L$ and $\sigma_2 = R$, after deliberation $M'_2, 1 \models B_2 R$

Majority Merge: Example 2

Step 1. Take the horizontal arrows

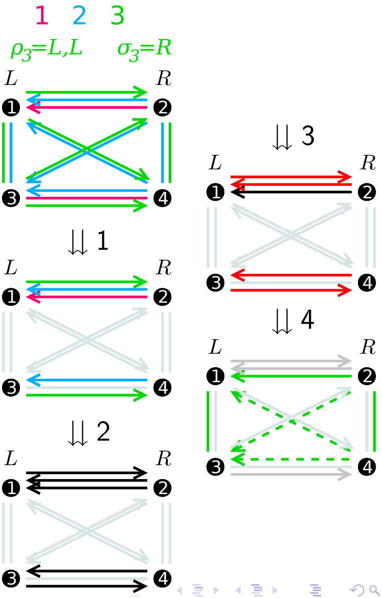
Step 2. Ignore their colors (mash them together)

Step 3. Remove conflicting pairs

Step 4. Restore information cell and establish preorder

Being given $\rho_3 = L, L$ and $\sigma_3 = R$, after deliberation $M_3', 1 \models B_3 L$
3 changes his mind after deliberation!

Based on M_3', 3 chooses his action based on a *decision rule*.



Decision Rules

If decided:

If the agent thinks he should go φ, he goes φ:

$$\text{If } M_i', 1 \models B_i \varphi \text{ then } \alpha_i = \varphi$$

What if he is indifferent?

If $M_i', 1 \models \neg B_i L \wedge \neg B_i R$ then $\alpha_i = \mathit{default}$

default rules:

- $\alpha_i = \sigma_i$ (used here for naïve agents)
- Randomize over $\{L, R\}$ (used here for open-minded agents)
- $\alpha_i = \alpha_{i-1}$
- Play mixed strategy with probabilities given by $\alpha_1, \ldots, \alpha_{i-1}$
- ...

Tipping Point

Tipping Point: Define action α_i to be a *tipping point* in the (partial) string

$$\sigma_{<i+1} = \alpha_1 \ldots \alpha_{i-1}\alpha_i$$

if, and only if, for all possible *consistent* strings

$$\sigma_{i>} = \alpha_{i+1} \ldots \alpha_n$$

extending $\sigma_{<i+1}$, it holds that for all

$$\alpha_j \in \sigma_{i>}, \alpha_j = \alpha_i$$

Consistent string: (loosely) a string if consistent if, given each agents interpretation of their public signals, they acted in accordance with the decision rules.

In Example 2, the string L, L, R is *not* be consistent.
If agent 3 had interpreted $\rho_3 = L, L$ differently so he was indifferent between L and R after deliberation and then randomized, then it would be.

Case 1: Naïve Agents

Interpretation Rule:

Where $j < i$ and $\varphi = \{L, R\}$

$$\text{if } \alpha_j = \varphi \text{ then } M_i, 1 \models B_i B_j \varphi$$

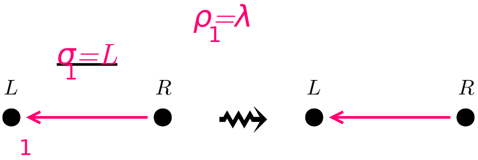
Default rule:

$$\text{if } M_i', 1 \models \neg B_i L \wedge \neg B_i R \text{ then } \alpha_i := \sigma_i$$

Private signals:

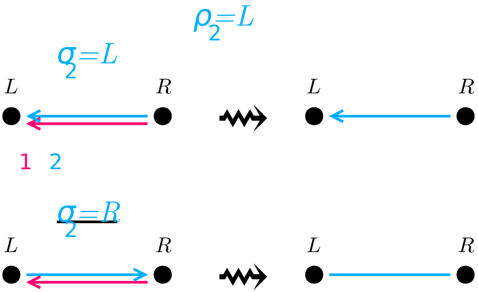
$$\sigma_1 = L$$
$$\sigma_2 = R$$
$$\sigma_3 = R$$
$$\sigma_4 = R$$
$$\sigma_5 = !$$

Case 1, Agent 1



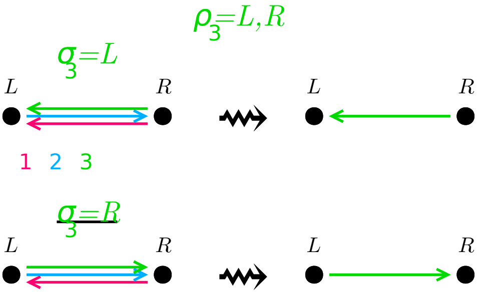
$M'_1, 1 \models B_1 L$ so $\alpha_1 = L$

Case 1, Agent 2



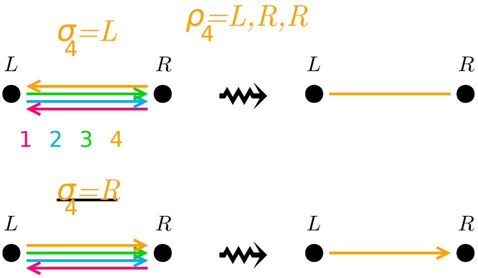
$M'_2, 1 \models \neg B_2 L \wedge \neg B_2 R$ so $\alpha_2 = default = \sigma_2 = R$

Case 1, Agent 3



$M'_3, 1 \models B_3 R$ so $\alpha_3 = R$

Case 1, Agent 4

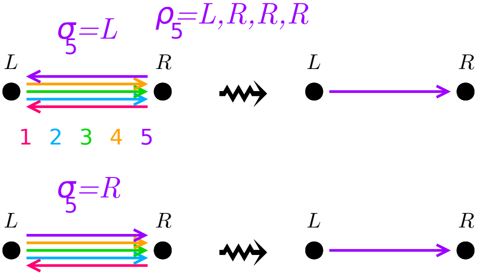


If $\sigma_4 = L$, then $M'_4, 1 \models \neg B_4 L \wedge \neg B_4 R$ so $\alpha_4 = \mathit{default} = \sigma_4 = L$

$\Rightarrow$ Back to agent 2 situation of indifference.

If $\sigma_4 = R$, then $M'_4, 1 \models B_4 R$ so $\alpha_4 = R$.

Case 1, Agent 5



No matter which signal 5 gets, $M'_5, 1 \models B_5 R$ so $\alpha_5 = R$

And so on for $i > 5$.

So $\alpha_4 = R$ was tipping point!

Conclusion, case 1

- There was a tipping point, being the action of agent 4: if $\alpha_4 = R$, then all subsequent agents would go right.
- General: if there are ***two more*** agents choosing one action than there are agents choosing the other action, a cascade will commence.
- This is caused by, amongst others, equal weights of the arrows and the naïve interpretation rule.

Case 2: Open-minded agents

Interpretation Rule:

Consider all possibilities consistent with private signal $\sigma_j \in \{L, \lambda, R\}$ for $j < i$, and the default rule.

(λ: Open-minded agents consider it possible that the agents before did not have an initial preference wrt. L and R)

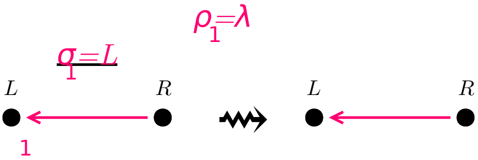
Default Rule:

if $M_i', 1 \models \neg B_i L \wedge \neg B_i R$ **then randomize over** $\{L, R\}$

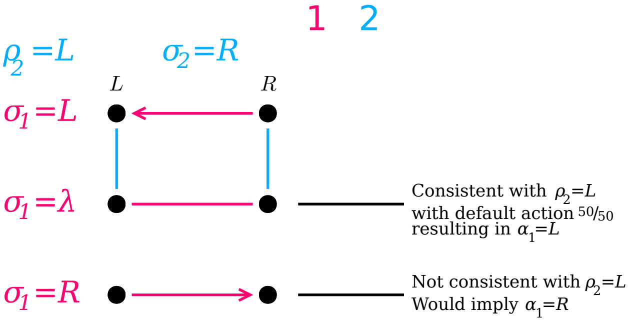
Private signals:

$$\sigma_1 = L$$
$$\sigma_2 = R$$
$$\sigma_3 = R$$
$$\sigma_4 = R$$
$$\sigma_5 = !$$

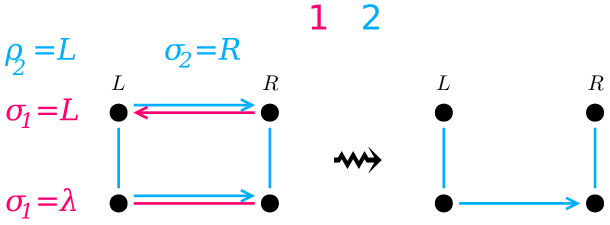
Case 2, Agent 1



Case 2 – what does 2 consider?



Case 2, Agent 2, $M_2 \rightsquigarrow M_2'$



$M_2', 1 \models B_2 R$, so $\alpha_2 = R$

Case 2: Agents 3, 4 and 5

Agent 3: 3 knows that $\sigma_2 = R$.

Otherwise, $\alpha_2 \neq R$ as 2 would be convinced by agent 1's action.

After deliberation, 3 believes R, so $\alpha_3 = R$.

Agent 4: 4 knows that $\sigma_3 \neq L$, but maybe $\sigma_3 = R$ or $\sigma_3 = \lambda$.

$\sigma_3 = L$ is inconsistent with $\rho_4 = L, R, R$

$\sigma_3 = R$ is consistent with $\rho_4 = L, R, R$

$\sigma_3 = \lambda$ is consistent with $\rho_4 = L, R, R$

3 was indifferent, randomized and played R.

After deliberation, 4 believes R, so $\alpha_4 = R$.

Case 2: Agents 3, 4 and 5

Agent 5: 5 cannot rule out anything!
$\sigma_4 = L$ is consistent with $\rho_5 = L, R, R, R$: 4 was indifferent, randomized and played R
$\sigma_4 = R$ is consistent with $\rho_5 = L, R, R, R$
$\sigma_4 = \lambda$ is consistent with $\rho_5 = L, R, R, R$: 4 was convinced by 2 and 3.

After deliberation: if $\sigma_5 = R$, then $B_5 R$. If $\sigma_5 = L$, then indifferent so $\alpha_5 = default$.

Agent 6: Cannot rule out anything, etc. and so on for $j > 6$.

Upshot: An agent that receives private signal L will divert to *default*, and hence possibly play L.
Therefore *there will always be a possible consistent string where for*

$$j > i,\ \alpha_j \neq \alpha_i$$

Hence, **there can be no tipping point**.

Conclusions

- Naïve agents will cause a cascade if *two more* choose one action over the other.
- Open-Minded agents will *never* cause a cascade.
  - Open-minded agents will not fall in the lemming trap: they will consider it possible that previous agents acted on a weak basis, and will therefore be indifferent after deliberation.
- **Overall:** considering higher-order information has an effect: considering more possibilities eliminates cascade.

Further perspectives

- 1st vs. 3rd person: all this is from the modelers point of view. What happens if we limit visibility so agents only perceive the last say 3 agents?
- How will knowledge or belief about interpretation and decision rules affect things?
- Other "lemming effect" scenarios:
  - Going to the party: two unknowing agents leading a flock astray.
  - Causing Jaywalkers: knowing agent leading unknowing agents across red light (and into traffic!)
  - ...